1. Introduction

The basic assumption to most of modern option valuation theory is that of a compounded binomial process. Essentially, over some (short) interval of time, the underlying asset’s value will change to one of two possible values. This is illustrated in the following diagram where the asset begins with a value of $S$, and at the end of the period attains a value of either $uS$ or $dS$. Because the option payoffs depend only on the asset price at time one (and exogenous parameters), the option will also follow a binomial process. This is illustrated in the second part of the diagram where the option moves from an original value of $O$ to a value of $O_u$ if the asset price increases, or $O_d$ if the asset decreases in value. One would need the particular option contract to assign values to $O_u$ and $O_d$.

Figure 1: A Security and its Option

2. Binomial Interpretations

2.1 Hedging:

Suppose we buy one share of stock and use $h$ options to hedge the uncertainty of the investment. That is, if possible we create a riskless investment using the asset and its option. The correct structure is one which would satisfy the following equations,

$$Su + hO_u = Y(1 + r)$$
$$Sd + hO_d = Y(1 + r)$$

where $Y$ is the present value of the equivalent riskless investment. The solution for $h$ and $Y$ are

$$Y = S \frac{O_d u - O_u d}{r(O_d - O_u)}$$
$$h = -\frac{dS - uS}{O_d - O_u}$$

Two investments which produce exactly the same cash flows must sell for the same value (The ‘law
of one price’). To prevent arbitrage, the value of the option must satisfy

\[ S + hO = Y \]

and, substituting from 1,

\[ O = O_u \frac{(r - d)}{r(u - d)} + O_d \frac{(u - r)}{r(u - d)} \]

2.2 State Pricing:

Suppose we have data concerning the stock price and the riskless rate. Suppose also that over the appropriate time interval there are only two possible ‘states of the world’. A state is a description of the economy specifying the prices of all economic quantities. The logic then runs that the value of a dollar in the state does not depend upon which security delivers it and neither does its present value. The way in which the market arrives at the present value of the stock and the present value of the riskless security is interpreted as present valuing the state dependent payoffs using state prices \( Q_u \) and \( Q_d \). Specifically

\[ S = uSQ_u + dSQ_d \]

\[ 1 = rQ_u + rQ_d \]

These are two equations in two unknowns and are solved for the state prices. The solutions are

\[ Q_u = \frac{(r - d)}{r(u - d)} \quad Q_d = \frac{(u - r)}{r(u - d)} \]  
(2)

and the value of the option is now computed using the state prices from

\[ O = Q_uO_u + Q_dO_d \]  
(3)

2.3 Risk Neutral Pricing.

Examining our previous results on state prices, it can be argued that these prices function like probabilities except that they don’t add to one. Looking at 2, it is true that

\[ rQ_u + rQ_d = 1 \]

and it is natural to define a ‘risk-neutral’ probability \( (p) \), as

\[ p = rQ_u \quad 1 - p = rQ_d \]

then 1 or 3 are reinterpreted as

\[ O = \frac{pO_u + (1 - p)O_d}{r} \]

This last equation is interpreted as computing the expected value of end-of-period option value (using the risk neutral probabilities) and then discounting at the riskless rate. This approach would only be appropriate (assuming the probabilities were accurate) if the investor did not require a premium for risk (i.e. if he/she were risk-neutral). In practice the message is that fudging the probabilities correctly can sidestep the risk issues associated with options and other derivatives.