Notes on State Pricing

Fin 480

1. Introduction
This set of notes is intended as an introduction to the Time–State Preference Model (hereafter, TSP). This model is now considered the basic model in the theory of finance, and has, somewhat surprisingly, found its most widespread application in real world option valuation. The model is introduced at this point in the course because of its facility for clarifying the insights and structure of theoretical arguments in finance.

The basic concepts associated with this model are:

- **States**: At each future date, , the economy can be in one (and only one) of states. A state is a complete description of the economy: cash flows from securities, their market values, and the prices of all goods, are known. One way of thinking about states is that each represents one of the possible future equilibria for the economy. The consumer/investors in this model know at the characteristics and prices associated with all future states.

- **State Prices**: State prices are the present value factors for a claim paying $1 if the state occurs at time .

We concentrate on single period state prices in this note, but show how multiperiod valuation techniques can be built from them.

2. The Binomial State Model, Valuation and State Prices
The simplest form of the TSP Model assumes that two states of the world exist at the end of the period. Thus, for a security which represents a claim to the end of period cash flows, , the following figure applies: where indicates the end of period cash flow from the security if the (somewhat arbitrarily labelled) ‘up’ state occurs, and is the corresponding result in the ‘down’ state.

Valuation in the TSP Model rests on the observation that the present value of $1 to be received if a certain state occurs is independent of which security delivers it and, therefore, so is the present value of the claim to that dollar. Literally, if the up state occurs at time one, $1 provided by IBM will have the same value as $1 provided by Xerox, and the present value of the claim to that dollar must have been the same at time zero.

---

1 The model is sometimes called the Arrow/Debreu Model after its originators — Kenneth Arrow and Gerard Debreu.
Suppose we intend to value the claim portrayed in Figure 1. Let $Q_u$ and $Q_d$ denote the respective state prices prevailing at time zero. These respective prices are associated with the following basic securities.

\[
\begin{align*}
T = 0 & \\
& Q_u \
& 0 \\
& 1 \\

T = 1 & \\
& Q_d \\
& 1 \\
& 0
\end{align*}
\]

Figure 2: Basic Securities

is the present value of the first diagram above, and $Q_d$ is the present value of the second diagram above. The TSP Model calculates the present value of the random income stream of Figure 1 as

\[
V_x = Q_u X_u + Q_d X_d
\]

This last equation reflects the TSP conclusion that the present value of a security is simply the sum of its state priced payoffs. It is motivated by the observation that purchasing $X_u$ units of the ‘up’ basic security and $X_d$ units of the ‘down’ basic security (Figure 3) would provide exactly the same payoffs, and cost $Q_u X_u + Q_d X_d$.

To prevent arbitrage, the first security must sell at exactly the same price.

3. Calculation and Interpretation of State Prices

Using state prices to value claims is fairly easy. The important questions would seem to be how to obtain state prices, and just what do they represent? Obtaining state prices, in theory, also is simple. Suppose we know that there are only two states of the world at the end of the period, and that a security which conforms to Figure 1 sells currently for $100 and has $X_u + V_u = 150$, and $X_d + V_d = 60$. At the same time, a riskless asset offers a 11.111\% rate of return over the period. Both assets must conform to the state pricing of equation (2);

\[
100 = 150 Q_u + 80 Q_d \\
1 = 1.11Tq_u + 1.11Tq_d \\
\]

The solution to these simultaneous equations is,

\[
Q_u = 0.50 \\
Q_d = 0.40 \\
\]

In general, to solve for $n$ state prices, we need pricing information on $n$ (linearly independent) securities ($n$ equations in unknowns).

Finally, what do the levels of these state prices represent? By state pricing, the value of a claim to $1$ is the sum of the state prices.

\[
\frac{1}{1 + R_f} = Q_u 1 + Q_d 1 = Q_u + Q_d \\
\]

Hence, the sum of the state prices represents the present value of a riskless dollar. In a more complete development of the model than that given here, one finds that state prices are proportional to the product of the marginal utility of a $1$ in the state and the probability of the state occurring. For states where marginal utility is high (scarcity), the state price is high. For states with low probability, the state price is low (ceteris paribus).

4. Applications to Options:

A basic assumption to most of modern option valuation theory is that the underlying asset has returns which are generated by a (short run) binomial process. Essentially, over some (short) interval of time, the underlying asset’s value will change to one of two possible values. This is illustrated in the following diagram.
where the asset begins with a value of $S$, and at the end of the period attains a value of either $uS$ or $dS$. Because the option payoffs depend only on the asset price at time one (and exogenous parameters), the option will also follow a binomial process. This is illustrated in the second part of the diagram where the option moves from an original value of $O$ to a value of $O_u$ if the asset price increases, or $O_d$ if the asset decreases in value. One would need the particular option contract to assign values to $O_u$ and $O_d$, unknowns.

A particular model often used in the option pricing literature specifies the future stock prices in terms of an ‘up’ and a ‘down’ factor, and determines the state prices from these quantities. For notational simplicity, let $r$ denote the quantity $1 + R_f$. The two equations determining state prices may now be written as

\[ S = uSQ_u + dSQ_d \]
\[ 1 = rQ_d + rQ_d \]

Solving the last two equations algebraically, we obtain the formulae for the state prices:

\[ Q_u = \frac{r-d}{r(u-d)} \]  
\[ Q_d = \frac{u-r}{r(u-d)} \]

These formulae tell us, among other things, that state prices depend only upon the pattern of returns. Particular values of $u$, $d$, and $r$, will allow us to evaluate state prices ($u > r > d$). Finally, a particular option contract would specify values for $O_u$ and $O_d$. The state prices would then allow us to price the security using the formula

\[ O = O_uQ_u + O_dQ_d \]

because the present value of the payoffs depends only upon the magnitude of payoff in each state and the respective state prices, and is otherwise independent of the security delivering those payoffs.

As an example, let $S = 100$ denote the initial price of the security. Let $uS = 150$ denote the value of the stock in the ‘up’ state. Then $u = 1.50$. Similarly, let $dS = 80$, then $d = .80$. Let the riskless rate be 11.111% ($r = 1.1111T$). Finally, suppose we wish to price a one period put option with an exercise price of $100$. Our two securities appear as in figure 4 below. Using, these last two equations, we obtain $Q_u = .40$, $Q_d = .50$, as the state prices by substituting $r = 1.1111T$, $u = 1.50$, and $d = .80$. The value of the put is therefore $10.00 (P = .4 \times 0 + .5 \times 20)$. The reader should verify that a one period call with an exercise price of 100 would have a value of $20$. These values are consistent with put-call parity.

5. Multiperiod Option Valuation

Suppose we are interested in pricing a European option which will last for two periods — specifically, a two period put with an exercise of $100$. The general theory accommodates multiperiod present value, but our one period approach provides all the necessary apparatus. We begin with a depiction of the security price movement over the two period. The following diagram is that depiction and assumes that ‘up’ and ‘down’ factors remain the same in the second period.

So far, we only know how to value one period prior to the cash flows. Let us begin the process at time one assuming that the stock has risen to $150$. Figure 6 diagrams this setting. Clearly the put is worthless under
these conditions as the diagram conveys the fact that no payoff can be expected from the put. Letting \( P_u \) denote the value of that put, \( O_u = 0 \). Suppose the value of the stock is $80 at time one. Then the appropriate diagram is Figure 7. and the put is worth \( O_d = .4 \times 0 + .5 \times 36 = $16 \). We are now in a position to note that if we purchase the put at time zero we will have a security which is worthless \( (S_u = 150) \), or worth $18 \( (S_d = 80) \). This is portrayed in Figure 9 Utilizing our state prices, the initial value of the European option must be $9.00 If we value the two period European call \( (X = 100) \), its values are \( (C_u = O_u = $60, C_d = O_d = $88, C_0 = O = $28) \)

You may have noticed that an odd thing happened in this last example. The two period put was actually worth less than the one period put. This can't happen with calls, but it can happen for the put and indicates that the put should have been prematurely exercised. To see this lets price an American put with the same exercise price and two periods to maturity. We begin in the same way. The put is worthless at time one if the stock price is $150. If the stock price is $80, the put is worth $18 if held for the last period. Because the put is American (exercisable at any time), we observe that we would make $20 if it is exercised now \( (S_d = 80, \text{ payoff } = 100 - 80 = 20) \). Going back to time 0, this put is worth $10 (the same as the one period put) because it will be optimal to exercise it at time 1 or it will be worthless.

To summarize this methodology:

- Begin at the next to last date (time 1 in our two period example).
- For each possible value of the underlying asset, determine the payoffs to holding the security for the remaining period and the value of the option at the beginning of that period.
- If the option is American, compute the value of premature exercise, and use the larger of this value, or that from the previous part.
- Move backwards on period at a time until you arrive at time 0. The result is the option value.

6. Problems

1. A risky security sells for $40. Next period this security will be worth either $60 or $24. The riskless rate of interest is 10%.

   a. What are the state prices implicit in the pricing of these securities?
   b. Suppose the probabilities of each state are .6 and .4, respectively, what is the discount rate associated with the risky security?
   c. Compute the futures price associated with a one period futures contract. What relationship does this bear to the current spot price?

2. Using the data of problem 1,

   a. Compute the value of a one period call with an exercise price of $40.
   b. What is the expected rate of return and return distribution for this option?
   c. Compute the value of a one period put with an exercise price of $40.
d. What is the return distribution and expected rate of return on this option?
e. Check to see if put-call parity holds.

3. Suppose the risky security’s return pattern repeats in the second period (up 50% or down 40%), and the riskless rate is again 10% (state prices remain the same).
   a. What is the value of a two period, European call exercisable at $40.
   b. What is the value of a two period, American call exercisable at $40.
   c. What is the value of a two period, European put exercisable at $40.
   d. What is the value of a two period, American put exercisable at $40.
   e. Check put-call parity for the American options.
   f. Check put-call parity for the European options.

4. Use the data of the previous problems, but assume that the stock will pay a $4 dividend at time 1.
   a. What is the value of a two period, European call exercisable at $40.
   b. What is the value of a two period, American call exercisable at $40.
   c. What is the value of a two period, European put exercisable at $40.
   d. What is the value of a two period, American put exercisable at $40.
   e. Check put-call parity for the American options.
   f. Check put-call parity for the European options.
Value $T = 1$ \quad Value $T = 2$ \quad Value $T = 1$ \quad Value $T = 2$

\[
\begin{align*}
150 & \quad 225 & \quad 0 & \quad 0 \\
120 & & & \\
\end{align*}
\]

Figure 6: The Stock the Put Option

Value $T = 1$ \quad Value $T = 2$ \quad Value $T = 1$ \quad Value $T = 2$

\[
\begin{align*}
80 & \quad 120 & \quad 0 & \quad 36 \\
64 & & \quad 0 & \\
\end{align*}
\]

Figure 7: The Stock the Put Option

Initial Value \quad Value $T = 1$ \quad Initial Value \quad Value $T = 1$

\[
\begin{align*}
100 & \quad 150 & \quad 0 & \quad 16 \\
80 & & \quad 16 & \\
\end{align*}
\]

Figure 8: The Stock the Put Option