Research Description and Plans (Byung-Jay Kahng)

In "non-commutative [or quantum] geometry", C^*-algebras take the roles of (continuous functions on) non-commutative topological spaces or non-commutative manifolds. Meanwhile, jumping from ordinary geometry to non-commutative geometry is broadly called a "quantization" process. Here, the framework for the classical setting is Poisson geometry.

My main research interests have been focused on these two aspects [non-commutative geometry and quantization], with special attention paid to the topics relating to locally compact quantum groups. In the next a few years, I plan to continue my research in this direction, emphasizing the roles being played by Poisson geometry in operator algebras.

1. Deformation quantization. C^*-algebraic quantum groups.

Given a Poisson manifold M, "deformation quantization" is a process of deforming the pointwise multiplication on $C^\infty(M)$ to a noncommutative one (usually given as a formal power series in the deformation parameter), in such a way that its first-order approximation with respect to the parameter is given by the Poisson bracket on M [3]:

$$\frac{1}{i\hbar}(f \times_\hbar g - g \times_\hbar f) \longrightarrow \{f, g\}, \quad \text{as } \hbar \to 0.$$

In more analytical settings, Rieffel introduced the notion of "strict deformation quantization" [28], [30]. Here, the deformed function algebra is required to be a C^*-algebra and the deformation is described in terms of a continuous field of C^*-algebras (with C^*-norms changing continuously). More general versions have appeared since [31], [19], [24].

In particular, I have been interested in the problem of deforming certain non-linear Poisson brackets. Let us be a little more specific. In [7], I defined a class of Poisson brackets that can be regarded as "cocycle perturbations" of linear Poisson brackets (which exist on dual vector spaces of Lie algebras). Given such a Poisson bracket, I find a certain group cocycle which is then used to define a twisted group C^*-algebra. The method is reasonably general, and under suitable conditions, I can show that this C^*-algebra provides a deformation quantization of the given Poisson structure (A similar result was obtained by Landsman [19].).

We may ask whether we can generalize this result, for instance to the setting of Poisson structures on Lie algebroids using the framework of groupoid C^*-algebras. At least for linear Poisson structures, some work has been done by Ramazan [26]; but so far, not much progress has been made on non-linear Poisson structures. Remembering the modest success I had with my class of Poisson brackets (on certain Poisson–Lie groups) giving rise to some specific quantum groups (see the paragraphs below), it will be interesting to see if one can do similar things in the setting of this more general class of Poisson brackets on Lie algebroids, possibly obtaining quantum groupoids.

On the other hand, my main motivation for considering the aforementioned class of Poisson brackets was to develop practical methods of constructing specific quantum groups. Indeed, my construction of the example given in [8] was based on deforming a Poisson–Lie group equipped with a compatible non-linear Poisson bracket. This example in [8] is a non-compact, C^*-algebraic quantum group, which may be viewed as a "quantum Heisenberg group algebra" (later, a more precise description of the example was given in [11]). More examples are given in [10], [14].

The framework of deformation quantization only gives us a quantized space. So to obtain quantum groups, we need extra tools like "multiplicative unitary operators" (in the sense of Baaj and Skandalis [1]). But the key strategy is that we can use the geometric data at the classical level of Poisson–Lie groups to relate with the information at the quantum level. For instance, important quantum group structure maps like comultiplication, antipode, and Haar weight usually can be predicted by the data at the Poisson level.

The C^*-algebraic quantum groups we study are *locally compact quantum groups* ([18], [23]), whose topological aspects are due to our C^*-algebra approach. These are generally considered to be technically more difficult than the ones based on purely algebraic framework of quantized universal enveloping algebras [5], [27]. In addition, unlike the case of compact quantum groups ([40], [39]), there have been only a small number of non-compact examples constructed so far.

Since the more well-known method of "generators and relations" (like the one used in [40]) is not very useful in non-compact settings due to the unboundedness of generators, our strategy of using Poisson geometry for guidance is particularly valuable in constructing non-compact quantum groups. See [14] for a refined approach of obtaining a class of quantum groups, relying less on the actual deformation quantization process, but nonetheless taking full advantage of the information at the Poisson–Lie group level to construct the quantum groups. The method is simple, but broad enough to include many of the earlier known non-compact quantum groups ([36], [34], [29]) as special cases.

So far, I have been working with the frameworks of twisted group C^*-algebras or "twisted crossed product algebras" (which are generalizations of group C^*-algebras with a nontrivial group cocycle as well as a nontrivial group action incorporated [25]), as the underlying C^*-algebras of the quantum group examples. This is the case for the constructions done in [8], [11], [10], [14]. Presumably, as a generalization, we may introduce the framework of "partial crossed products" (by Exel [6]) or the (even more general) framework of "C^*-correspondences" associated with certain Hilbert bimodules [15]. They will correspond to a larger class of (non-linear) Poisson brackets. If successful, it is reasonable to speculate that we could construct in this way a larger class of (non-compact, solvable) quantum groups.

2. Representation theory and Poisson geometry.

The interplay between Poisson geometry and quantum groups becomes more prominent when we study the representation theory of quantum groups.

Let us first note that the study of ordinary group representation theory is enriched using Kirillov's orbit theory [16], where the irreducible unitary representations of a Lie group are related with the orbits under the coadjoint action of the group. In the case of quantum groups, the unitary representations are related with the orbits under the "dressing action". But it is known that the dressing orbits are exactly the symplectic leaves of the underlying Poisson geometric structure [21]. The quantum group orbit analysis is a still on-going endeavor, which began around the early 90's [33], [20], [17].

Some work in this direction for our specific quantum group has already begun [9], [12]. In addition to pointing out the one-to-one correspondence between irreducible representations and the dressing orbits (which turns out to be a topological homeomorphism), we can also discuss the decomposition of a given representation as a direct integral of irreducible representations (in particular, for the representations obtained by restricting or by inducing) and can describe the inner tensor products of representations via Clebsch–Gordan type decompositions. A Plancherel type theorem also exists.

As a next step, I wish to conduct the orbit analysis from the point of view of the quantum group structure theory, that is, the study of subalgebras and ideals of our Hopf C^*-algebra (quantum group), which would be quantum homogeneous spaces and quantum subgroups. In cases of some specific compact quantum groups, like the quantum $SU(2)$, the study of the quantum homogeneous spaces turned out to be quite fruitful [31], [32]. In our case, we wish to develop our approach by trying to gain insights from Poisson geometric data. For instance, I obtained in [12] some C^*-algebras that are considered as quantizations of the dressing orbits. Noting that the dressing orbits are themselves homogeneous spaces, it will be interesting to investigate further the "quantized dressing orbits" from this viewpoint.

Among the tools that were useful in studying the representation theory was the "(quasitriangular) quantum universal R-matrix" type operator I found in [8]. With this operator, I could show that our quantum group has the following "quasitriangular" property:

> **Theorem.** For any two *-representations π and ρ of our quantum group (A, Δ), the quantum Heisenberg group algebra, their inner tensor products "commute" (i. e. $\pi \boxtimes \rho \cong \rho \boxtimes \pi$). However, the intertwining operators $T_{\pi\rho}$ and $T_{\rho\pi}$ (from $\pi \boxtimes \rho$ to $\rho \boxtimes \pi$, and vice versa) are in general not inverses of each other (i. e. We can have: $T_{\pi\rho}T_{\rho\pi} \neq \mathrm{Id}$.).

This is a genuinely quantum behavior that does not occur in ordinary group representation theory. And, this "quasitriangularity" is getting a lot of attention from various areas of mathematics and physics, because of their applications to braids and knot theory as a quantum symmetry (See [4] for a discussion.). Meanwhile, let us point out that the quantum R-matrix has a counterpart called the "classical" r-matrix, at the Poisson–Lie group (more precisely, Lie bialgebra) level [8]. We again observe an interplay between classical/geometric data and the property at the quantum level.

3. Quantum double construction.

One of the most celebrated of Hopf algebra constructions is the quantum double construction of Drinfeld [5]. It is expressed as $D(A) = \hat{A}^{\mathrm{op}} \bowtie A$, given by mutual actions between a Hopf algebra A and its dual $\hat{A}$ (with opposite multiplication). The original definition was for finite-dimensional cases, but a natural generalization exists as a double crossed product [22], [42], [2]. The result is a new Hopf algebra $D(A)$ which is quasitriangular. Even a simple case like $D(G)$, the quantum double of the group algebra of an ordinary group G, turns out to be quite useful in some physics applications.

The quantum double construction for our specific example is carried out in [13] (The multiplicative unitary operators are quite useful here. See also the general constructions in [1], [42], [2].). But I wish to further explore various problems concerning the quantum double, as described below.

One important project is to make a deeper connection between our quantum situation and Poisson geometry. The quantum group A, its dual $\hat{A}$, and the quantum double $D(A)$ correspond nicely with the dual pair of Poisson–Lie groups and the double Poisson–Lie group, or the Manin triple [21]. But the latter is exactly the setting where the dressing actions and dressing orbits arise. Remembering from previous section that dressing orbits play a significant role in orbit theoretic approach to representation theory, we see the value of further exploring this interconnection.

Another direction of research involves two-sided actions between A and $\hat{A}$, which arise in the definition of $D(A)$. Even though the use of multiplicative unitary operators is helpful, having to work with two-sided actions is rather cumbersome. In the case of $D(G)$, however, this is not a problem: It is known that $D(G)$ can be identified (as a C^*-algebra) with $C_0(G) \rtimes_\alpha G$, where the (one-sided) action α of G on $C_0(G)$ is the conjugate action.

The reason behind this phenomenon is that the group algebra of G, or $C^*(G)$, is cocommutative. In general, a quantum group (A, Δ) is usually non-cocommutative and the two-sided actions do not necessarily degenerate into a one-sided action. Nevertheless, if A is quasitriangular (associated with its quantum R-matrix operator), the R-matrix makes it "almost-cocommutative" [4], in the sense that for $a \in A$, we have: $R\Delta(a)R^{-1} = \Delta^{\mathrm{cop}}(a)$. Thus it is reasonable to ask for a result like the following:

> **Problem.** For a Hopf C^*-algebra (or quantum group) (A, Δ) which satisfies the quasitriangular property, its quantum double $D(A) = \hat{A}^{\mathrm{op}} \bowtie A$ can be realized as:
>
> $$D(A) \cong B \rtimes A,$$

with one-sided action (Here B may be different from $\hat{A}$ and could be quite complicated to describe.).

Majid actually has a result of this type, although his is in the purely algebraic framework. I wish to explore if similar results can be obtained in the framework of C^*-algebras and *-isomorphisms. If successful, it will be quite helpful in the development of structure theory and representation theory of the quantum double. Moreover, having the underlying C^*-algebraic structure of a crossed product algebra gives us an added advantage that it is readily viewed as a quantized space, while it is at the same time group-like, being a quantum group.

Meanwhile, at least in the case of $A = C^*(G)$, I can show that there exists a natural twisting of the coalgebra structure on $D(A)$, which in turn provides a twisting of the algebra structure on $\widehat{D(A)}$. The resulting C^*-algebra (though not a Hopf C^*-algebra) happens to coincide with the Weyl algebra: $C_0(G) \rtimes_\tau G \cong \mathcal{K}\big(L^2(G)\big)$, where τ is the translation. A similar result has been known for some time (since early 90's) at the purely algebraic level, but so far no generalizations were carried out in the C^*-algebra framework. One of my current projects is to try to extend the above situation to the case of general locally compact quantum groups: It will presumably involve twisting of the quantum double $\hat{A}^{\mathrm{op}} \bowtie A$ and its dual object, and possibly obtaining the C^*-algebra $A\hat{A}^{\mathrm{op}}$, which will no longer be isomorphic to $\mathcal{K}(\mathcal{H})$ and can even be non-type I.

4. Haar weights and theoretical aspects of quantum groups.

At this stage, the general theory of (non-compact) locally compact quantum groups is still not quite settled, although significant progress has been made recently (See [41], [18], [23].). Among the most serious obstacles in the operator algebra approach is the point that the operation of multiplication, $m : A \odot A \to A$, cannot be continuously extended to the C^*-algebra $A \otimes A$. This, in turn, makes it difficult to literally adopt the purely algebraic definition of the antipode (coinverse). At least in the case of compact quantum groups [39], Woronowicz and others could get around this problem by working with the notion of Haar functional (which is shown to exist) and a certain density condition (corresponding to the cancellation property of a group).

In the general (non-compact) case, so far there is no existence theorem for Haar weights. Because of this, in any approach to the general theory, the notion of Haar weights and the assumption of their existence play a central role. They are also closely linked to the notion of multiplicative unitary operators ([1], [41]), which are among the indispensable tools in both the theory and examples.

In [11] and [10], I carried out some discussions on Haar weights for our specific examples. The multiplicative unitaries also play prominent roles. Ours are rather simple situations, but we still have to work with non-tracial

Haar weights or non-unimodular Haar weights. Moreover, most of the techniques being used there (originally due to Van Daele [37], [38]) are not very type-specific.

In addition to being the central ingredient to the quantum group theory, study of Haar weights on locally compact quantum groups leads to a wide variety of problems, especially in connection with the (generalized, Pontryagin-type) duality theory. Some of them are due to the fact that the Haar weights are approximately KMS and are related with certain modular automorphism groups. Also, as we see in [38], some interesting connections exist with topics like Heisenberg commutation relations and commuting squares.

Meanwhile, there is a somewhat different approach to handle the obstacles mentioned above. By working with the notion of Haagerup tensor products (from operator space theory, developed by Effros, Ruan, and others) and by considering $A \otimes_h A$, one can remedy the problem of continuously extending the multiplication. It is true that a Haagerup tensor product norm $\| \, \|_h$ is not a C^*-norm, but the new framework still does allow us to treat various Hopf algebraic statements with the language of operators and non-commutative geometry/topology. The research in this direction is being initiated by Vaes and Van Daele [35], which still carries several open questions relating the theory of Hopf C^*-algebras (or locally compact quantum groups) and that of operator spaces.

For instance, consider the "multiplier Haagerup tensor product" $A \otimes_{mh} A$ (as defined in [35]). It is the space of pairs of maps (ρ_1, ρ_2) from A to $A \otimes_h A$ satisfying $(a \otimes 1)\rho_1(b) = \rho_2(a)(1 \otimes b)$ for all $a, b \in A$. The multiplier Haagerup tensor product plays an important role in developing the notion of Hopf C^*-algebras, and we certainly have: $A \otimes_h A \subseteq A \otimes_{mh} A$. On the other hand, it is still not quite clear what the characterizing properties are (other than the definition above) for the elements in $A \otimes_{mh} A$.

My main attention so far has been geared more toward the interplay between Poisson geometry and C^*-algebraic quantum groups, but I am also very much interested in broadening my research topics to include these more theoretic aspects of Haar weights and Hopf C^*-algebras, which will give us a deeper connection with other topics in operator algebras like factors and commuting squares, or the ones in operator space theory.

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