COVERINGS, LAPLACIANS, AND HEAT KERNELS OF DIRECTED GRAPHS

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ABSTRACT. Combinatorial covers of graphs were defined by Chung and Yau ([3]). Their main property is that they preserve the spectra of their Laplacians. We extend their definition to directed graphs. As an application, we compute the spectrum of the normalized Laplacian of the homesick random walk RW_μ on the line. Using this calculation, we show that the heat kernel on the weighted line can be computed from the heat kernel of '$(1 + 1/\mu)$-regular' tree.

1. INTRODUCTION

Chung and Yau defined a variation of the Laplacian on an undirected weighted graph, called the *normalized Laplacian.* For regular graphs the spectrum of the adjacency operator on the graph can be recovered from the spectrum of the normalized Laplacian. Also, they gave the definition of a combinatorial cover of graphs ([3]) and they showed that certain combinatorial covers preserve the spectrum of the normalized Laplacians. They used this method to calculate the spectrum of infinite trees, lattices and certain k-regular graphs. Also, they used their machinery to calculate the heat kernel of the graphs involved. The main idea is that if a graph G covers a graph H, then H has a simpler combinatorial structure than G and thus its spectrum can be computed more easily than that of G. In [2], the definition of the normalized Laplacian was extended to directed graphs and its main properties were investigated.

Our main example of directed graphs is derived from homesick random walks on graphs. Homesick random walks on Cayley graphs were used in [4], where a connection between the homesick parameter and the growth of the group was given. This is related to the question asked by M. Gromov:

Question. *What is the relation between the spectrum of a random walk on the Cayley graph of the group to the geometry at infinity of the group?*

In this paper, we extend the definition of the combinatorial cover to directed weighted graphs and we prove the analogues of the properties in [3]. In particular, we show that combinatorial covers preserve the spectrum of the normalized Laplacians, in certain cases. Furthermore, we prove the analogues of the spectral properties and heat kernels in [3] for directed graphs.

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Then we apply our methods to compute the spectrum of the combinatorial Laplacian of the homesick random walk defined on the infinite line. More precisely, we construct a cover from the homesick random walk on the line to a weighted ray. Notice that normalized Laplacians were defined for finite directed graphs. So, we first approximate the weighted ray by the weighted segment of finite length l, we compute its spectral properties and we let $l \to \infty$. Using these results, we compute the heat kernel of the infinite length, which is closely related to heat kernel of the $1 + 1/\mu$-regular tree.

Theorem (Main Theorem)**.** *For the homesick random walk with parameter μ on the infinite line, the heat kernel $H_t(a,b)$ satisfies:*

$$(1)\ \ H_t(a,0) = \frac{\mu\sqrt{2(\mu+1)}}{\pi}\int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos x)}\sin x\left[\frac{1}{\mu}\sin(a+1)x - \sin(a-1)x\right]}{\mu^2+2\mu+1-4\mu\cos^2 x}dx.$$

(2) *If both a and b are not 0, then $H_t(a,b)$ is:*

$$\pm\frac{2}{\pi}(\mu^2+\varepsilon)\int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos x)}[\sin((|a|+1)x) - \sin(|a|-1)x][\sin((|b|+1)x) - \sin(|b|-1)x]}{\mu^2+2\mu+1-4\mu\cos^2 x}dx,$$

where $\varepsilon = sign(a)\cdot sign(b)$.

In the appendix, we realize the homesick random walk on the k-regular tree as a cover over a weighted ray. Using approximation by finite segments, we get some information about the eigenvalues of the normalized Laplacian.

The methods used suggest that if the underlying graph has enough symmetries, then the theory of normalized Laplacians can be extended to infinite graphs. This will be pursued in a forthcoming paper.

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2. PRELIMINARIES

We consider a weighted directed graph G which has a vertex set $V = V(G)$ and a weight function $w : V \times V \to \mathbb{R}$ such that

$$w(u,v) \geq 0, \text{ for all } u, v \in V.$$

For $u, v \in V$, if $w(u,v) > 0$, then we say (u,v) is an edge and u is *adjacent* to v. The degree d_v of a vertex v is defined as:

$$d_v = \sum_{u \in V} w(v,u).$$

Our main example of weighted directed graphs will be the lazy random walks used in [4]. Given a simple undirected unweighted based graph (G, z) we weight the edge (u, v) as follows:

$$w(u,v) := \begin{cases} \dfrac{\mu}{d_u + (\mu-1)d_u^-}, & v \in S_{|u|-1}(z) \\ \dfrac{1}{d_u + (\mu-1)d_u^-}, & \text{otherwise} \end{cases}$$

where d_u is the degree of u and $d_u^- = |N(u) \cap S_{|u|-1}(z)|$, where $S_r(z)$ denotes the combinatorial ball of radius r centered at z. That defines a random walk on G denoted RW_μ. In [4], it was shown that when the graph is the Cayley graph of a finitely generated group G and the base point is the identity then:

- If $\mu < \mathrm{gr}(G)$, then RW_μ is transient.
- If $\mu > \mathrm{gr}(G)$, then RW_μ is positive recurrent

We will use covers to study the spectrum of a random walk.

Definition 2.1. Let $\tilde{G}$ and G be two weighted directed graphs. We say $\tilde{G}$ is a *covering* of G (or G is covered by $\tilde{G}$) if there is a map $\pi : V(\tilde{G}) \to V(G)$ satisfying the following two properties:

(i) There is an $m \in \mathbb{R}$, called the *index* of π; such that for $u, v \in V(G)$, we have

$$\sum_{\substack{x\in\pi^{-1}(u)\\ y\in\pi^{-1}(v)}} w(x,y) = mw(u,v), \quad \text{and} \quad \sum_{\substack{x\in\pi^{-1}(u)\\ y\in\pi^{-1}(v)}} w(y,x) = mw(v,u).$$

(ii) For $x, y \in V(\tilde{G})$ with $\pi(x) = \pi(y)$ and $v \in V(G)$, we have

$$\sum_{z\in\pi^{-1}(v)} w(z,x) = \sum_{z'\in\pi^{-1}(v)} w(z',y), \quad \text{and} \quad \sum_{z\in\pi^{-1}(v)} w(x,z) = \sum_{z'\in\pi^{-1}(v)} w(y,z').$$

Remark 2.2.

(1) Notice that the definition generalizes the definition given in [3]. We need to consider "two sided" sums because the graphs are directed.

(2) The definition of covering given above does not correspond to the classical definition of graph coverings. In this paper, a cover will mean a covering as above.

As in [3], it is not hard to show the following properties of covers.

Lemma 2.3. *Suppose $\tilde{G}$ is a covering of G with index m. Let $u, v \in V(G)$ and $x \in \pi^{-1}(v)$, then*

(i)

$$\sum_{z\in\pi^{-1}(u)} w(x,z) = \frac{m}{|\pi^{-1}(v)|} w(v,u);$$

(ii)

$$\sum_{z\in V(\tilde{G})} w(x,z) = \frac{m}{|\pi^{-1}(v)|} \sum_{u\in V(G)} w(v,u);$$

(iii)

$$d_x = \frac{m}{|\pi^{-1}(v)|} d_v.$$

From the definition of a covering (i) and (ii) still hold if we replace $w(x,z)$ by $w(z,x)$ and $w(v,u)$ by $w(u,v)$.

From now on, we concentrate on finite graphs. The transition probability matrix P of a graph is defined as

$$P(u,v) = \frac{w(u,v)}{d_u}.$$

It is obvious that $P(u,v) > 0$ only if (u,v) is an edge. Further as in the undirected case

$$\sum_v P(u,v) = 1,$$

however in the directed case it is not true in general that $\sum_u P(u,v) = 1$. The transition probability matrix defines a *random walk* on the graph.

The Perron-Frobenius Theorem ([5]) implies that the transition probability matrix P of a graph has a unique left eigenvector ϕ with $\phi(v) > 0$ for all v, and $\phi P = \phi$. We will treat ϕ as a row vector. We can normalize and choose ϕ such that

$$\sum_v \phi(v) = 1.$$

We call ϕ the *Perron vector* of P. We can now define the *Laplacian* $\mathcal{L}$ of a directed graph ([2]):

$$\mathcal{L} = I - \frac{\Phi^{1/2} P \Phi^{-1/2} + \Phi^{-1/2} P^* \Phi^{1/2}}{2}$$

where Φ denotes the diagonal matrix with entries $\Phi(v,v) = \phi(v)$ and P^* is the conjugated transpose of P. The Laplacian satisfies $\mathcal{L}^* = \mathcal{L}$, that is, the Laplacian is Hermitian. The *spectrum of a graph* is the eigenvalues and eigenfunctions of the Laplacian of the graph.

2.1. **Example - The k-regular Tree.** Consider the homesick random walk on the k-regular tree T_k with root z. In particular the weights are given as follows:

$$w(x,y) = \begin{cases} \dfrac{1}{k}, & x = z, y \sim z, \\ \dfrac{1}{k+\mu-1}, & x \neq z, y \notin S_{|x|-1}(z), \\ \dfrac{\mu}{k+\mu-1}, & x \neq z, y \in S_{|x|-1}(z) \end{cases}$$

The combinatorial l-sphere centered at z has $s_l = k(k-1)^{l-1}$ if $l > 0$. We write

$$x_i^{(l)}, \ \ i = 1, 2, \cdots, s_l,$$

for the elements of the combinatorial l-sphere centered at z. Let $T_k^{(l)}$ be the l-ball in T_k i.e., we truncate T_k to include all the vertices within combinatorial distance l from z. We consider the homesick random walk induced on $T_k^{(l)}$. It has the same weights as the random walk on T_k except that:

$$w(x_i^{(l)}, y) = 1, \text{ when } x_i^{(l)} \sim y.$$

Let $P^{(l)}$ be the matrix of the random walk. Also, set

$$\rho = \frac{k-1}{\mu}.$$

Lemma 2.4. *Let ψ_l be the Frobenius–Perron vector of $T_k^{(l)}$. Then*

$$\psi_l(v_i^{(m)}) = \psi_l(v_i^{(m)}), \text{ for all } 0 \le m \le l, \;\; 1 \le i, j \le s_m.$$

Thus ψ_l is constant along each sphere centered at z. Furthermore,

$$\psi_l(v_i^{(n)}) = \begin{cases} \dfrac{1-\rho}{2(1-\rho^l)}, & n = 0 \\ \dfrac{(1+\rho)(1-\rho)}{2\mu^{n-1}k(1-\rho^l)}, & 1 \le n \le l-1 \\ \dfrac{1-\rho}{2\mu^{l-1}k(1-\rho^l)}, & n = l. \end{cases}$$

Proof. In the process of calculating the Frobenius–Perron eigenfunction we will show that its values are constant on the spheres of $T_k^{(l)}$. For simplicity, we set:

$$\alpha = \frac{\mu}{k+\mu-1}, \;\; \beta = \frac{1}{k+\mu-1}$$

For any $x_i^{(l)}$ there is a single vertex $x_j^{(l-1)}$ that is adjacent to it. The equation $\psi P^{(l)} = \psi$ implies

$$\beta\psi(x_j^{(l-1)}) = \psi(x_i^{(l)}) \text{ for each } x^{(l)} \sim y^{(l-1)}.$$

For a vertex $x_j^{(l-1)}$, using the previous equation,

$$\beta\psi(x_m^{(l-2)}) + (k-1)\psi(x_i^{(l)}) = \psi(x_j^{(l-1)}), \;\; x_m^{(l-2)} \sim x_j^{(l-1)}, \; x_i^{(l)} \sim x_j^{(l-1)}.$$

That implies

$$\beta\psi(x_m^{(l-2)}) = (1 - \beta(k-1))\psi(x_j^{(l-1)}).$$

Solving for $\psi(x_j^{(l-1)})$, we get

$$\psi(x_j^{(l-1)}) = \frac{1}{\mu}\psi(x_m^{(l-2)}).$$

Thus for two vertices in S_{l-1} have a common neighbor in S_{l-2} then ψ has the same value. For the next step notice that

$$\beta\psi(x_n^{(l-3)}) + (k-1)\alpha\psi(x_j^{(l-1)}) = \psi(x_m^{(l-2)}), \;\; x_n^{(l-3)} \sim x_m^{(l-2)}, \; x_j^{(l-1)} \sim x_m^{(l-2)}.$$

As before simplifying,

$$\psi(x_m^{(l-2)}) = \frac{1}{\mu}\psi(x_n^{(l-3)})$$

Continuing like that we get that

$$\psi(x_m^{(t)}) = \frac{1}{\mu}\psi(x_n^{(t-1)}),\ \ x_m^{(t)} \sim x_n^{(t-1)},\ \ l-2 \leq t \leq 2.$$

For the vertices of distance 1 from z,

$$\frac{1}{k}\psi(z) + (k-1)\alpha\psi(x_i^{(2)}) = \psi(x_j^{(1)}),\ \ x_i^{(2)} \sim x_j^{(1)}.$$

Thus

$$\frac{1}{k}\psi(z) = \left(1 - \frac{(k-1)\alpha}{\mu}\right)\psi(x_j^{(1)}) \implies \psi(x_j^{(1)}) = \frac{1}{k\alpha}\psi(z) = \frac{k+\mu-1}{\mu k}\psi(z)$$

Therefore ψ has the same value on S_1 and thus ψ is constant on each sphere. Inductively, we get that

$$\psi_l(v_i^{(n)}) = \begin{cases} \dfrac{k+\mu-1}{\mu^n k}\psi_l(z), & 1 \leq n \leq l-1 \\ \dfrac{1}{\mu^{l-1}k}\psi_l(z), & n = l. \end{cases} = \begin{cases} \dfrac{1+\rho}{\mu^{n-1}k}\psi_l(z), & 1 \leq n \leq l-1 \\ \dfrac{1}{\mu^{l-1}k}\psi_l(z), & n = l. \end{cases}$$

Adding up all the terms:

$$\begin{aligned} \sum_{i,n}\psi_l(v_i^{(n)}) &= \left(\frac{\mu+k-1}{\mu}\sum_{i=0}^{l-2}\rho^i + \rho^{l-1} + 1\right)\psi_l(z) = \left(\frac{(\rho+1)(1-\rho^{l-1})}{1-\rho} + \rho^{l-1} + 1)\right)\psi_l(z) \\ &= \left(\frac{2(1-\rho^l)}{1-\rho}\right)\psi_l(z) \end{aligned}$$

So the normalization condition implies:

$$\psi_l(z) = \frac{1-\rho}{2(1-\rho^l)},$$

and the Frobenius–Perron vector is given by:

$$\psi_l(v_i^{(n)}) = \begin{cases} \dfrac{1-\rho}{2(1-\rho^l)}, & n = 0 \\ \dfrac{(1+\rho)(1-\rho)}{2\mu^{n-1}k(1-\rho^l)}, & 1 \leq n \leq l-1 \\ \dfrac{1-\rho}{2\mu^{l-1}k(1-\rho^l)}, & n = l. \end{cases}$$

□

Remark 2.5. Notice that the limit, as $l \to \infty$ of $\psi_l(z)$ is non-zero iff $\rho < 1$. That happens iff the random walk is positive recurrent ([4]). If this is the case,

$$\psi(z) = \lim_{l\to\infty} \psi_l(z) = \frac{1-\rho}{2}.$$

Furthermore, taking limits as $l \to \infty$, we get the following formula for a candidate for the Frobenius–Perron vector for the homesick random walk on T_k:

$$\psi_l(v_i^{(n)}) = \begin{cases} \dfrac{1-\rho}{2}, & n = 0 \\ \dfrac{(1+\rho)(1-\rho)}{2\mu^{n-1}k}, & n \geq 1. \end{cases}$$

A direct calculation shows that ψ is the Frobenius–Perron vector for T_k, when $\rho < 1$.

Let Ψ_l be the diagonal matrix with entries $\psi_l(v_i^{(n)})$. Then, for two adjacent vertices,

$$\Psi_l^{1/2} P_l \Psi^{-1/2}(v_i^{(n)}, v_j^{(n)}) = \Psi_l^{-1/2} P_l^* \Psi^{1/2}(v_i^{(n)}, v_j^{(n)}) = \begin{cases} \dfrac{1}{\sqrt{k(\rho+1)}}, & (i,j) \in \{(0,1),(1,0)\}, \\ \dfrac{1}{\sqrt{\mu}(\rho+1)}, & |i-j| = 1, 1 \leq i,j \leq l-1, \\ \dfrac{1}{\sqrt{\mu(\rho+1)}}, & (i,j) \in \{(l,l-1),(l-1,l)\}. \end{cases}$$

The the combinatorial Laplacian is given by:

$$L^{(l)}(v_i^{(n)}, v_j^{(m)}) = \begin{cases} 1, & v_i^{(n)} = v_j^{(m)}, \\ -\dfrac{1}{\sqrt{k(\rho+1)}}, & (n,m) \in \{(0,1),(1,0)\}, \\ -\dfrac{1}{\sqrt{\mu}(\rho+1)}, & |n-m| = 1, 1 \leq n,m \leq l-1, v_i^{(n)} \sim v_j^{(m)} \\ -\dfrac{1}{\sqrt{\mu(\rho+1)}}, & (n,m) \in \{(l,l-1),(l-1,l)\}, \\ 0, & \text{if the vertices are not adjacent.} \end{cases}$$

3. The spectrum of a graph and its coverings

We will show that there is a connection between the eigenvalues of a graph and a graph that covers it, however to begin we establish a connection between the respective Perron vectors. Again, all the graphs are finite.

Proposition 3.1. *Suppose G is a weighted directed graph with Perron vector ϕ and $\tilde{G}$ is a covering of G with index m with respect to the mapping π. Then the Perron vector of $\tilde{G}$ $\tilde{\phi}$ can be defined by*

$$\tilde{\phi}(x) = \frac{\phi(v)}{|\pi^{-1}(v)|}, \text{ for each } v = \pi(x).$$

Proof. It is enough to show that $\tilde{\phi}\tilde{P} = \tilde{\phi}$, where $\tilde{P}$ is the transition probability matrix of $\tilde{G}$. We make liberal use of Lemma 2.3. Suppose $v \in V(G)$ and $\pi(x) = v$.

$$\begin{aligned}
(\tilde{\phi}\tilde{P})(x) &= \sum_{y\in\tilde{G}} \tilde{\phi}(y)\tilde{P}(y,x) && = \sum_{u\in V(G)} \sum_{y\in\pi^{-1}(u)} \tilde{\phi}(y)\tilde{P}(y,x) \\
&= \sum_{u\in V(G)} \sum_{y\in\pi^{-1}(u)} \frac{\phi(u)}{|\pi^{-1}(u)|}\frac{w(y,x)}{d_y} && = \sum_{u\in V(G)} \frac{\phi(u)}{|\pi^{-1}(u)|} \sum_{y\in\pi^{-1}(u)} \frac{|\pi^{-1}(u)|}{md_u} w(y,x) \\
&= \sum_{u\in V(G)} \frac{\phi(u)}{md_u} \sum_{y\in\pi^{-1}(u)} w(y,x) && = \sum_{u\in V(G)} \frac{\phi(u)}{md_u} \frac{m}{|\pi^{-1}(x)|} w(u,v) \\
&= \frac{1}{|\pi^{-1}(v)|} \sum_{u\in V(G)} \phi(u)\frac{w(u,v)}{d_u} && = \frac{1}{|\pi^{-1}(v)|}(\phi P)(v) \\
&= \frac{1}{|\pi^{-1}(v)|}\phi(v) && = m\tilde{\phi}(x).
\end{aligned}$$

Note that the third to last equality follows from

$$\sum_{u\in V(G)} \phi(u)\frac{w(u,v)}{d_u} = \sum_{u\in V(G)} \phi(u)P(u,v) = \phi(v).$$

□

When computing the spectrum of a graph it is sometimes convenient to consider *harmonic eigenfunctions*. Let g denote an eigenfunction of $\mathcal{L}$ associated with the eigenvalue λ then $f = g\Phi^{-1/2}$ is called the harmonic eigenfunction. In [2], it was shown that:

$$\lambda f(v)\phi(v) = \frac{1}{2}\sum_{u}(f(v) - f(u))(\phi(u)P(u,v) + P(v,u)\phi(v)), \text{ for each } v \in V(G),$$

where f is a harmonic eigenfunction associated with the eigenvalue λ.

As in [3], using the above identity we get the following:

Lemma 3.2. *If $\tilde{G}$ is a covering of G, then an eigenvalue of G is an eigenvalue of $\tilde{G}$.*

Proof. If $\tilde{G}$ is a covering of G with respect to the map π of index m, we can lift the harmonic eigenfunction f of G (and the associated eigenvalue λ) to $\tilde{G}$ by defining, for each vertex x in $\tilde{G}$, $f(x) = f(v)$ where $v = \pi(x)$. We then have

$$\tfrac{1}{2}\sum_{y}(f(x) - f(y))(\tilde{\phi}(y)\tilde{P}(y,x) + \tilde{P}(x,y)\tilde{\phi}(x)) \quad =$$

$$\begin{aligned}
&= \tfrac{1}{2}\sum_y (f(x)-f(y))\left(\tilde{\phi}(y)\tfrac{w(y,x)}{d_y} + \tfrac{w(x,y)}{d_x}\tilde{\phi}(x)\right)\\
&= \tfrac{1}{2}\sum_u \sum_{z\in\pi^{-1}(u)} (f(v)-f(u))\left(\tfrac{\phi(u)}{|\pi^{-1}(u)|}\tfrac{w(z,x)}{d_z} + \tfrac{w(x,z)}{d_x}\ \tfrac{\phi(v)}{|\pi^{-1}(v)|}\right)\\
&= \tfrac{1}{2}\sum_u (f(v)-f(u))\left(\tfrac{\phi(u)}{|\pi^{-1}(u)|}\sum_{z\in\pi^{-1}(u)}\tfrac{w(z,x)}{d_z} + \tfrac{\phi(v)}{|\pi^{-1}(v)|}\sum_{z\in\pi^{-1}(u)}\tfrac{w(x,z)}{d_x}\right)\\
&= \tfrac{1}{2}\sum_u (f(v)-f(u))\left(\tfrac{\phi(u)}{|\pi^{-1}(u)|}\sum_{z\in\pi^{-1}(u)}\tfrac{|\pi^{-1}(u)|w(z,x)}{md_u} + \tfrac{\phi(v)}{|\pi^{-1}(v)|}\sum_{z\in\pi^{-1}(u)}\tfrac{|\pi^{-1}(v)|w(x,z)}{md_v}\right)\\
&= \tfrac{1}{2}\sum_u (f(v)-f(u))\left(\tfrac{\phi(u)}{md_u}\sum_{z\in\pi^{-1}(u)} w(z,x) + \tfrac{\phi(v)}{md_v}\sum_{z\in\pi^{-1}(u)} w(x,z)\right)\\
&= \tfrac{1}{2}\sum_u (f(v)-f(u))\left(\tfrac{\phi(u)}{md_u}\tfrac{m}{|\pi^{-1}(v)|}w(u,v) + \tfrac{\phi(v)}{md_v}\tfrac{m}{|\pi^{-1}(v)|}w(v,u)\right)\\
&= \tfrac{1}{|\pi^{-1}(v)|}\tfrac{1}{2}\sum_u (f(v)-f(u))(\phi(u)P(u,v)+P(v,u)\phi(v))\\
&= \tfrac{1}{|\pi^{-1}(v)|}\lambda f(v)\phi(v) \;=\; \lambda f(x)\tilde{\phi}(x).
\end{aligned}$$

□

Following the line of arguments in [3], we have the following:

Lemma 3.3. *Suppose $\tilde{G}$ is a covering of G with respect to the mapping π of index m. If a harmonic eigenfunction f of $\tilde{G}$, associated with an eigenvalue λ, has a nontrivial image in G, then λ is also an eigenvalue for G.*

Proof. For each $x \in \pi^{-1}(v)$,

$$\sum_y (f(x)-f(y))(\tilde{\phi}(y)\tilde{P}(y,x)+\tilde{P}(x,y)\tilde{\phi}(x)) = \lambda f(x)\tilde{\phi}(x).$$

By summing over x in $\pi^{-1}(v)$, we have

$$\sum_{x\in\pi^{-1}(v)}\sum_y (f(x)-f(y))(\tilde{\phi}(y)\tilde{P}(y,x)+\tilde{P}(x,y)\tilde{\phi}(x)) = \lambda \sum_{x\in\pi^{-1}(v)} f(x)\tilde{\phi}(x).$$

We define the induced mapping of f in G, denoted $g : V(G) \to \mathbb{R}$ by

$$g(v) = \sum_{x\in\pi^{-1}(v)} \frac{f(x)\tilde{\phi}(x)}{\phi(v)} = \frac{1}{|\pi^{-1}(v)|}\sum_{x\in\pi^{-1}(v)} f(x).$$

It is clear from the definition of g that

$$\lambda \sum_{x\in\pi^{-1}(v)} f(x)\tilde{\phi}(x) = \lambda g(v)\phi(v).$$

Now consider the following

$$\frac{1}{2}\sum_{x\in\pi^{-1}(v)}\sum_{y}(f(x)-f(y))(\tilde{\phi}(y)\tilde{P}(y,x)+\tilde{P}(x,y)\tilde{\phi}(x)). \tag{1}$$

We break the sum into two parts:

(i)

$$\begin{aligned}
\tfrac{1}{2}\sum_{x\in\pi^{-1}(v)} f(x)\sum_{y}(\tilde{\phi}(y)\tilde{P}(y,x)+\tilde{P}(x,y)\tilde{\phi}(x)) &= \tfrac{1}{2}\sum_{x\in\pi^{-1}(v)} f(x)(\tilde{\phi}(x)+\tilde{\phi}(x)) \\
&= \sum_{x\in\pi^{-1}(v)} f(x)\tilde{\phi}(x) \\
&= \tfrac{1}{2}g(v)(\phi(v)+\phi(v)) \\
&= \tfrac{1}{2}\sum_{u} g(v)(\phi(u)P(u,v)+P(v,u)\phi(v)).
\end{aligned}$$

(ii)

$$\begin{aligned}
\tfrac{1}{2}\sum_{x\in\pi^{-1}(v)}\sum_{y} f(y)(\tilde{\phi}(y)\tilde{P}(y,x)+\tilde{P}(x,y)\tilde{\phi}(x)) &= \\
&= \tfrac{1}{2}\sum_{u}\sum_{y\in\pi^{-1}(u)} f(y)\sum_{x\in\pi^{-1}(v)}(\tilde{\phi}(y)\tilde{P}(y,x)+\tilde{P}(x,y)\tilde{\phi}(x)) \\
&= \tfrac{1}{2}\sum_{u}\sum_{y\in\pi^{-1}(u)} f(y)\sum_{x\in\pi^{-1}(v)}\left(\tfrac{\phi(u)w(y,x)}{|\pi^{-1}(u)|d_y}+\tfrac{w(x,y)\phi(v)}{d_x|\pi^{-1}(v)|}\right) \\
&= \tfrac{1}{2}\sum_{u}\sum_{y\in\pi^{-1}(u)} f(y)\left(\tfrac{\phi(u)}{md_u}\sum_{x\in\pi^{-1}(v)} w(y,x)+\tfrac{\phi(v)}{md_v}\sum_{x\in\pi^{-1}(v)} w(x,y)\right) \\
&= \tfrac{1}{2}\sum_{u}\sum_{y\in\pi^{-1}(u)} f(y)\left(\tfrac{\phi(u)w(u,v)}{d_u|\pi^{-1}(u)|}+\tfrac{\phi(v)w(v,u)}{d_v|\pi^{-1}(u)|}\right) \\
&= \tfrac{1}{2}\sum_{u} g(u)(\phi(u)P(u,v)+P(v,u)\phi(v)).
\end{aligned}$$

We can now write expression (1) as

$$\frac{1}{2}\sum_{u}(g(v)-g(u))(\phi(u)P(u,v)+P(v,u)\phi(u)).$$

Hence,

$$\frac{1}{2}\sum_{u}(g(v)-g(u))(\phi(u)P(u,v)+P(v,u)\phi(u))=\lambda g(v)\phi(v).$$

If g is nontrivial, then λ is an eigenvalue of G. □

Definition 3.4. A graph $\tilde{G}$ is a *regular covering* of G if for a fixed vertex v in $V(G)$ and for any vertex x of $V(\tilde{G})$, $\tilde{G}$ is a covering of G under a mapping π_x which maps x into v. If π_x^{-1} is just

x then $\tilde{G}$ is a *strong regular covering.* Further, a graph G is said to be *distance regular* if G is a strong regular covering of a (weighted) path.

Remark 3.5. Strong regular coverings do not appear as often in weighted directed graphs. The reason is that the definition requires some degree of homogeneity. Usually, this type of homogeneity is lacking in the lazy random walks considered.

Lemma 3.6. *Suppose $\tilde{G}$ is a strong regular covering of G. Then, $\tilde{G}$ and G have the same eigenvalues.*

Proof. For any nontrivial harmonic eigenfunction f of $\tilde{G}$ we can choose v to be a vertex with nonzero value of f. The induced mapping of f in G has a nonzero value at v and therefore is a nontrivial harmonic eigenfunction for G. From Lemma 3.3, we see that any eigenvalue of $\tilde{G}$ is an eigenvalue of G. By Lemma 3.2, we conclude that $\tilde{G}$ and G have the same eigenvalues. □

3.1. **Example - The k-regular Tree as a Cover.** As in [3], we will realize T_k as a cover over a weighted ray. Let $\mathcal{P}^+$ be the weighted ray with $V(\mathcal{P}^+) = \mathbb{N}$ and

$$w(i,j) = \begin{cases} 1, & \text{if } i = 0,\ j = 1, \\ \dfrac{k\mu(k-1)^{i-1}}{\mu+k-1}, & \text{if } i > 0,\ i - j = 1 \\ \dfrac{k(k-1)^{i}}{\mu+k-1}, & \text{if } i > 0, i - j = -1 \end{cases}$$

As in [3],

$$\pi : T_k \to \mathcal{P}^+, \ \ \pi(x) = d(x, z),$$

where the distance is the combinatorial distance in T_k. The fact that π is a combinatorial cover of index 1 is proved as in [3].

This means that the eigenfunctions of the tree are the same as those of the path. Thus we set out to determine the eigenfunctions of the path.

First we must find the Laplacian of $\mathcal{P}^+$. To do that, let $\mathcal{P}^{(l)}$ be truncated path with

$$V(\mathcal{P}^{(l)}) = \{0, 1, \ldots, l\}$$

and all the weights as in $\mathcal{P}^+$ except that we set $w(l, l-1) = 1$. We will calculate the Laplacian of $\mathcal{P}^{(l)}$ and then we will take $l \to \infty$. The probability matrix is given by:

$$P_l = \begin{pmatrix} 0 & 1 & 0 & \cdots & \\ \frac{1}{1+\rho} & 0 & \frac{\rho}{1+\rho} & \cdots & \\ 0 & \frac{1}{1+\rho} & 0 & \frac{\rho}{1+\rho} & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

The Frobenius–Perron vector satisfies

$$(\phi_l(0), \phi_l(1), \phi_l(2), \ldots, \phi_l(l))P_l = (\phi_l(0), \phi_l(1), \phi_l(2), \ldots, \phi_l(l))$$

Thus the vector ϕ_l satisfies:

$$\phi_l(i) = \begin{cases} \rho^{i-1}(1+\rho)\phi_l(0), & 1 \leq i \leq l-1 \\ \rho^{l-1}\phi_l(0), & i = l. \end{cases}$$

$$\Phi_l^{\frac{1}{2}} P_l \Phi_l^{-\frac{1}{2}}(i,j) = \Phi_l^{-\frac{1}{2}} P_l^* \Phi_l^{\frac{1}{2}}(i,j) = \begin{cases} \dfrac{1}{\sqrt{1+\rho}}, & (i,j) = (0,1),\ (1,0) \\ \dfrac{\sqrt{\rho}}{1+\rho}, & |i-j| = 1,\ 2 \leq i \leq l-2 \\ \dfrac{\sqrt{\rho}}{1+\rho}, & (i,j) = (1,2),\ (l-1, l-2) \\ \sqrt{\dfrac{\rho}{1+\rho}}, & (i,j) = (l-1, l),\ (l, l-1) \end{cases}$$

Hence,

$$\mathcal{L}^{(l)}(i,j) = \begin{cases} 1, & i = j \\ -\dfrac{1}{\sqrt{1+\rho}}, & (i,j) = (0,1),\ (1,0) \\ -\dfrac{\sqrt{\rho}}{1+\rho}, & |i-j| = 1,\ 2 \leq i \leq l-2 \\ -\dfrac{\sqrt{\rho}}{1+\rho}, & (i,j) = (1,2),\ (l-1, l-2) \\ -\sqrt{\dfrac{\rho}{1+\rho}}, & (i,j) = (l-1, l),\ (l, l-1) \end{cases}$$

So if we set $\sigma = 1 + \rho$, then we get:

$$\mathcal{L}^{(l)}(i,j) = \begin{cases} 1; & i = j, \\ -\sqrt{\dfrac{1}{\sigma}}; & (i,j) = (0,1) \text{ or } (1,0), \\ -\dfrac{\sqrt{\sigma-1}}{\sigma}; & |i-j| = 1,\ 0 < i,j < l, \\ -\sqrt{\dfrac{\sigma-1}{\sigma}}; & (i,j) = (l-1,l) \text{ or } (l, l-1), \\ 0; & \text{otherwise.} \end{cases}$$

Note that this is exactly the same Laplacian as in [3] Section 5. Hence, the eigenvalues of $\mathcal{L}_l$ are 0, 2 and

$$\lambda_n = 1 - \frac{2\sqrt{\sigma-1}}{\sigma}\cos\frac{\pi n}{l} = 1 - \frac{2\sqrt{\mu(k-1)}}{\mu+k-1}\cos\frac{\pi n}{l}, \quad \text{for } n = 1, \ldots, l-1.$$

The eigenfunction ϕ_0 associated with the eigenvalue 0 can be written as $f_0/\|f_0\|$ where

$$\begin{aligned} f_0(0) &= 1, \\ f_0(p) &= \sqrt{\sigma(\sigma-1)^{p-1}}, \text{ for } 1 \leq p \leq l-1, \\ f_0(l) &= \sqrt{(\sigma-1)^{l-1}}. \end{aligned}$$

We now consider the eigenfunction ϕ_l which corresponds to the eigenvalue 2. In this case, $\phi_l = f_l/\|f_l\|$, where f_l is defined as follows:

$$\begin{aligned} f_l(0) &= 1, \\ f_l(p) &= (-1)^p\sqrt{\sigma(\sigma-1)^{p-1}}, \text{ for } 1 \leq p \leq l-1, \\ f_l(l) &= (-1)^l\sqrt{(\sigma-1)^{l-1}}. \end{aligned}$$

For each $n \in \{1, 2, \ldots, l-1\}$ there is an eigenfunction ϕ_n associated with the eigenvalue μ_n. We can write $\phi_n = f_n/\|f_n\|$ where

$$\begin{aligned} f_n(0) &= \sqrt{\frac{\sigma}{\sigma-1}}\sin\frac{\pi n}{l}, \\ f_n(p) &= \sin\frac{\pi n(p+1)}{l} - \frac{1}{\sigma-1}\sin\frac{\pi n(p-1)}{l}, \text{ for } 1 \leq p \leq l-1, \\ f_n(l) &= \frac{\sqrt{\sigma}}{\sigma-1}\sin\frac{\pi n}{l}. \end{aligned}$$

For $n = 1, \ldots, l-1$,

$$\|f_j\|^2 = \frac{l\sigma^2}{2(\sigma-1)^2}\left(1 - \frac{4(\sigma-1)}{\sigma^2}\cos^2\frac{n\pi}{l}\right).$$

Let $L^2(T_k^{(l)})$ be the L^2-space generated by the vertices of $T_k^{(l)}$. Then

$$L^2(T_k^{(l)}) = V_1 \oplus V_2$$

where is the direct sum of eigenspaces that contain eigenfunctions that not vanish at z. The second summand is its complementary subspace i.e., the subspace containing the eigenspaces that all eigenfunctions vanish at z. By Lemma 3.3, the eigenfunctions that do not vanish at z are exactly the eigenfunctions of $P_+^{(l)}$. Using the same methods as in Lemma 2.4 and the fact about the eigenvalues from Lemma 3.3, if ϕ is any eigenfunction in V_1, then

$$\phi(v_i^{(n)}) = \phi_j^{(n)}.$$

4. The heat kernel of a graph and its covering

We extend the definition of the heat kernel ([1]) to directed graphs. Given a finite weighted directed graph G, the heat kernel h_t is defined for $t \geq 0$. It is the solution to the *heat equation*:

$$\frac{\partial h_t}{\partial t} = -\mathcal{L} h_t, \quad h_0 = I.$$

Then h_t can be expressed as

$$h_t = e^{-t\mathcal{L}} = \sum_{r=0}^{\infty} (-1)^r \frac{t^r \mathcal{L}^r}{r!} = \sum_i e^{-\lambda_i t} P_i$$

where P_i is the projection into the eigenspace corresponding to the eigenvalue λ_i of $\mathcal{L}$. If u, v are vertices of G, then

$$h_t(u,v) = \sum_i e^{-\lambda_i t} \psi_i(u) \psi_i(v)$$

where ψ_i ranges over the orthonormal eigenfunctions of $\mathcal{L}$ ([1], [3]).

As shown above, the eigenvalues of a graph G and $\tilde{G}$ a covering of G are related. Therefore, it should be expected that their respective heat kernels are related. Indeed they are as illustrated by the following lemmas, which are the directed analogues of the results in [3].

An r-*walk* on a graph is a sequence of vertices $p_r = (u_0, u_1, \ldots, u_r)$ so that (u_i, u_{i+1}) is an edge. The weight of p_r is defined as

$$\omega(p_r) = \prod_{i=0}^{r-1} \frac{(\Phi^{1/2} P \Phi^{-1/2} + \Phi^{-1/2} P^* \Phi^{1/2})(u_i, u_{i+1})}{2} = \prod_{i=0}^{r-1} \frac{\phi(u_i) P(u_i, u_{i+1}) + \phi(u_{i+1}) P(u_{i+1}, u_i)}{2\sqrt{\phi(u_i)\phi(u_{i+1})}}.$$

Lemma 4.1. *Suppose $\tilde{G}$ is a covering of G. Let $\tilde{h}_t$ and h_t denote the heat kernels of $\tilde{G}$ and G, respectively. Then*

$$\sum_{x \in \pi^{-1}(u)} \sum_{y \in \pi^{-1}(v)} \tilde{h}_t(x,y) = \sqrt{|\pi^{-1}(u)||\pi^{-1}(v)|}\, h_t(u,v).$$

Proof. A direct calculation shows that

$$h_t(u,v) = e^{-t} \sum_r S_r(u,v) \frac{t^r}{r!}$$

where S_r is the sum of weights of all r-walks joining u and v. We want to show that the total weights of the paths in $\tilde{G}$ lifted from p_r (i.e. whose image in G is p_r) is exactly the weight of p_r in G multiplied by $\sqrt{|\pi^{-1}(u_0)||\pi^{-1}(u_r)|}$. Let p_{r-1} denote the walk $u_0, \ldots, u_{r-1}$. Suppose $u_{r-1} \neq u_r$.

For each path $\tilde{p}_{r-1}$ lifted from p_{r-1}, its extensions to paths lifted from p_r has total weights

$$\begin{aligned}
& -w(\tilde{p}_{r-1}) \sum_{z\in\pi^{-1}(u_r)} \frac{\tilde{\phi}(\tilde{u}_{r-1})\tilde{P}(\tilde{u}_{r-1},z)+\tilde{\phi}(z)\tilde{P}(z,\tilde{u}_{r-1})}{2\sqrt{\tilde{\phi}(\tilde{u}_{r-1})\tilde{\phi}(z)}} \\
= \; & -w(\tilde{p}_{r-1}) \sum_{z\in\pi^{-1}(u_r)} \frac{\frac{\phi(u_{r-1})}{|\pi^{-1}(u_{r-1})|}\tilde{P}(\tilde{u}_{r-1},z)+\frac{\phi(u_r)}{|\pi^{-1}(u_r)|}\tilde{P}(z,\tilde{u}_{r-1})}{2\sqrt{\frac{\phi(u_{r-1})\phi(u_r)}{|\pi^{-1}(u_{r-1})||\pi^{-1}(u_r)|}}} \\
= \; & \frac{-w(\tilde{p}_{r-1})\sqrt{|\pi^{-1}(u_{r-1})||\pi^{-1}(u_r)|}}{2\sqrt{\phi(u_{r-1})\phi(u_r)}} \sum_{z\in\pi^{-1}(u_r)} \left(\frac{\phi(u_{r-1})}{|\pi^{-1}(u_{r-1})|}\tilde{P}(\tilde{u}_{r-1},z)+\frac{\phi(u_r)}{|\pi^{-1}(u_r)|}\tilde{P}(z,\tilde{u}_{r-1})\right)
\end{aligned}$$

We will return later to the last expression and now consider the two sums contained therein:

(i)

$$\begin{aligned}
\sum_{z\in\pi^{-1}(u_r)} \frac{\phi(u_{r-1})}{|\pi^{-1}(u_{r-1})|}\tilde{P}(\tilde{u}_{r-1},z) &= \frac{\phi(u_{r-1})}{|\pi^{-1}(u_{r-1})|} \sum_{z\in\pi^{-1}(u_r)} \frac{w(\tilde{u}_{r-1},z)}{d_{\tilde{u}_{r-1}}} \\
&= \frac{\phi(u_{r-1})}{d_{u_{r-1}}|\pi^{-1}(u_{r-1})|} w(u_{r-1},u_r) \\
&= \frac{\phi(u_{r-1})}{|\pi^{-1}(u_{r-1})|} P(u_{r-1},u_r).
\end{aligned}$$

(ii)

$$\begin{aligned}
\sum_{z\in\pi^{-1}(u_r)} \frac{\phi(u_r)}{|\pi^{-1}(u_r)|}\tilde{P}(z,\tilde{u}_{r-1}) &= \frac{\phi(u_r)}{|\pi^{-1}(u_r)|} \sum_{z\in\pi^{-1}(u_r)} \frac{w(z,\tilde{u}_{r-1})}{d_z} \\
&= \frac{\phi(u_r)}{md_{u_r}} \sum_{z\in\pi^{-1}(u_r)} w(z,\tilde{u}_{r-1}) \\
&= \frac{\phi(u_r)}{d_{u_r}|\pi^{-1}(u_{r-1})|} w(u_r,u_{r-1}) \\
&= \frac{\phi(u_r)}{|\pi^{-1}(u_{r-1})|} P(u_r,u_{r-1}).
\end{aligned}$$

Returning to the expression from before with the sums simplified we have:

$$\begin{aligned}
& \frac{-w(\tilde{p}_{r-1})\sqrt{|\pi^{-1}(u_{r-1})||\pi^{-1}(u_r)|}}{2\sqrt{\phi(u_{r-1})\phi(u_r)}} \left(\frac{\phi(u_{r-1})}{|\pi^{-1}(u_{r-1})|}P(u_{r-1},u_r)+\frac{\phi(u_r)}{|\pi^{-1}(u_{r-1})|}P(u_r,u_{r-1})\right) \\
= \; & -w(\tilde{p}_{r-1})\frac{\sqrt{|\pi^{-1}(u_r)|}}{\sqrt{|\pi^{-1}(u_{r-1})|}} \left(\frac{\phi(u_{r-1})P(u_{r-1},u_r)+\phi(u_r)P(u_r,u_{r-1})}{2\sqrt{\phi(u_{r-1})\phi(u_r)}}\right)
\end{aligned}$$

By summing over all $\tilde{p}_{r-1}$, we have

$$\sum_{x\in\pi^{-1}(u)} \sum_{y\in\pi^{-1}(v)} S_r(x,y) = \sqrt{|\pi^{-1}(u)||\pi^{-1}(v)|}S_r(u,v).$$

□

As a consequence of Lemma 4.1, we have

Corollary 4.2. *Suppose* $\tilde{G}$ *is a strong regular covering of* G*. Let* $\tilde{h}_t$ *and* h_t *denote the heat kernels of* $\tilde{G}$ *and* G *respectively. For* $x \in \pi^{-1}(u)$*, we have*

$$\sum_{y \in \pi^{-1}(v)} \tilde{h}_t(x, y) = \sqrt{\frac{|\pi^{-1}(v)|}{|\pi^{-1}(u)|}} h_t(u, v).$$

Corollary 4.3. *Suppose* G *is a distance regular graph which is a covering of a path* P *with vertices* $v_0, \dots, v_p$ *where* $p = D(G)$*, the diameter of* G*. Suppose* G *and* P *have heat kernels* $\tilde{h}_t$ *and* h_t *respectively. For any two vertices* x *and* y *in* G *with distance* $d(x, y) = r$*, we have*

$$\tilde{h}_t = \sqrt{|\pi^{-1}(v_r)|} h_t(v_0, v_r).$$

The following theorem is from [3]. The proof in the directed case follows exactly as in the undirected case and is offered here for the sake of completeness.

Theorem 4.4. *Suppose* $\tilde{G}$ *is a strong regular covering of* G*. Let* v *denote the vertex of* G *with preimage in* $\tilde{G}$ *consisting of one vertex. Then any eigenvalue* λ *of* $\tilde{G}$ *has multiplicity*

$$n \sum_i \frac{f_i^2(v)}{\|f_i\|^2},$$

where $n = |V(\tilde{G})|$ *and the* f_i*'s span the eigenspace of* λ *in* G*. If the eigenvalue* λ *has multiplicity* 1 *in* G *with eigenfunction* f*, then the multiplicity of* λ *in* $\tilde{G}$ *is*

$$\frac{n f^2(v)}{\|f\|^2}.$$

Proof. Suppose $\tilde{G}$ has heat kernel $\tilde{h}_t$ and G has heat kernel h_t. Since $\tilde{G}$ is a strong regular covering of G, we have

$$Tr(\tilde{h}_t) = \sum_{x \in V(\tilde{G})} \tilde{h}_t(x, x) = n h_t(v, v) = n \sum_j e^{-t\lambda_j} \frac{f_j^2(v)}{\|f_j\|^2}.$$

Therefore, the multiplicity of λ_j in $\tilde{G}$ is exactly

$$\frac{n f_j^2}{\|f_j\|^2}$$

if the multiplicity of λ in G is 1. In general, the multiplicity of λ in $\tilde{G}$ is

$$n \sum_i \frac{f_i^2(v)}{\|f_i\|^2}$$

where the f_i's span the eigenspace of λ in G. □

4.1. **The Heat Kernel of the k-regular Tree.** Let T_k be an in Example 2.1 and $\mathcal{P}_+$ be as in Example 3.1. Using the calculations in [3] we have that the heat kernel $h_t^{(l)}$ of $P^{(l)}$ satisfies

$$h_t^{(l)}(0,0) = \sum_{j=1}^{l-1} \frac{e^{-t(1-\frac{2\sqrt{\sigma-1}}{\sigma}\cos\frac{j\pi}{l})}\sin^2\frac{j\pi}{l}}{\frac{l\sigma}{2(\sigma-1)}\left(1-\frac{4(\sigma-1)}{\sigma^2}\cos^2\frac{j\pi}{l}\right)} + \frac{1}{\|f_0\|^2} + \frac{1}{\|f_l\|^2}.$$

When l approaches infinity, the heat kernel h_t of $\mathcal{P}_+$ satisfies:

$$h_t(0,0) = \frac{2\sigma(\sigma-1)}{\pi}\int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\sigma-1}}{\sigma}\cos x)}\sin^2 x}{\sigma^2-4(\sigma-1)\cos^2 x}dx.$$

In general, for $a \geq 1$, we have

$$h_t(0,a) = \frac{2\sqrt{\sigma(\sigma-1)}}{\pi}\int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\sigma-1}}{\sigma}\cos x)}\sin x[(\sigma-1)\sin(a+1)x-\sin(a-1)x]}{\sigma^2-4(\sigma-1)\cos^2 x}dx.$$

Let $L^2(T_k^{(l)})$ be the L^2-space generated by the vertices of $T_k^{(l)}$. Then

$$L^2(T_k^{(l)}) = V_1 \oplus V_2$$

where is the direct sum of eigenspaces that contain eigenfunctions that not vanish at z. The second summand is its complementary subspace i.e., the subspace containing the eigenspaces that all eigenfunctions vanish at z. Let $H_t^{(l)}$ denote the heat kernel of the subtree of T_k spanned by the l-ball centered at z. We write

$$H_t^{(l)} = H_{1,t}^{(l)} + H_{2,t}^{(l)}$$

where $H_{i,t}^{(l)}$ is the restriction to V_i. By Lemma 3.3, the eigenfunctions that do not vanish at z are exactly the eigenfunctions of $P_+^{(l)}$. Using the same methods as in Lemma 2.4 and the fact about the eigenvalues from Lemma 3.3, if ϕ is any eigenfunction in V_1, then

$$\phi(v_i^{(n)}) = \phi_j^{(n)}.$$

As in [3], we have that

Theorem 4.5. *With the above notation,*

$$H_{1,t}(0,v_i^{(n)}) = \frac{2}{\pi(\sigma-1)^{n/2-1}}\int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\sigma-1}}{\sigma}\cos x)}\sin x\,[(\sigma-1)\sin(n+1)x-\sin(n-1)x]}{\sigma^2-4(\sigma-1)\cos^2 x}dx.$$

For the term corresponding to V_2, et $v = (v_{\sigma_i^{(n)}})$ be an eigenvector of $L^{(l)}$ with $v_z = 0$. Let λ be its eigenvalue. Then the equation $vL_l = \lambda v$ implies:

$$
\begin{aligned}
\sum v_{\sigma_i^{(1)}} &= 0 \\
v_{\sigma_i^{(1)}} - \frac{1}{\sqrt{\mu}(\rho+1)} \sum_{\sigma_j^{(2)} \sim \sigma_i^{(1)}} v_{\sigma_j^{(2)}} &= \lambda v_{\sigma_i^{(1)}}, \\
-\frac{1}{\sqrt{\mu}(\rho+1)} v_{\sigma_j^{(n-1)}} + v_{\sigma_i^{(n)}} - \frac{1}{\sqrt{\mu}(\rho+1)} \sum_{\sigma_j^{(n+1)} \sim \sigma_i^{(n)}} v_{\sigma_j^{(n+1)}} &= \lambda v_{\sigma_i^{(n)}},\ \sigma_j^{(n-1)} \sim \sigma_i^{(n)},\ n < l-1, \\
-\frac{1}{\sqrt{\mu}(\rho+1)} v_{\sigma_j^{(l-2)}} + v_{\sigma_i^{(l-1)}} - \frac{1}{\sqrt{\mu(\rho+1)}} \sum_{\sigma_j^{(l)} \sim \sigma_i^{(l-1)}} v_{\sigma_j^{(l)}} &= \lambda v_{\sigma_i^{(l-1)}},\ \sigma_j^{(l-1)} \sim \sigma_i^{(l-1)},\ n = l-1, \\
-\frac{1}{\sqrt{\mu(\rho+1)}} v_{\sigma_j^{(l-1)}} + v_{\sigma_i^{(l)}} &= \lambda v_{\sigma_i^{(l)}},\ \sigma_j^{(l-1)} \sim \sigma_i^{(l-1)},\ n = l.
\end{aligned}
$$

The last equation implies that, if $\sigma_i^{(l)}$ and $\sigma_j^{(l)}$ are adjacent to the same vertex, then $v_{\sigma_i^{(l)}} = v_{\sigma_j^{(l)}}$. Also, each of the k-vertices $\sigma_i^{(n)}$ determines a subtree with root z. There are k such subtrees. It is clear from the form of the equations that:

(1) Each subtree determines exactly the same system. Thus we need to find the eigenvalues of the square matrix of side $(k-1)^{l-1} - 1$, Λ_l with entries indexed by the vertices in the subtree:

$$
\Lambda_l(v_{\sigma_i^{(n)}}, v_{\sigma_j^{(m)}}) = \begin{cases} 1, & \sigma_i^{(n)} = \sigma_j^{(m)} \\ \beta, & 1 \le i, j \le l-1, \sigma_i^{(n)} \sim \sigma_j^{(m)} \\ \gamma, & (i,j) \in \{(l, l-1), (l-1, l)\} \sigma_i^{(n)} \sim \sigma_j^{(m)} \\ 0, & \text{otherwise} \end{cases}
$$

where

$$
\beta = -\frac{1}{\sqrt{\mu}(\rho+1)}, \quad \gamma = -\frac{1}{\sqrt{\mu(\rho+1)}}.
$$

(2) The connection between the solutions coming from different branches is given by the fact that

$$
\sum_{i \in S(n)} v_{\sigma_i^{(n)}} = 0.
$$

5. THE HEAT KERNEL FOR THE ONE-DIMENSIONAL WEIGHTED LATTICE GRAPH

We consider the homesick random walk with parameter λ on the infinite path $\mathcal{P}$ (the one-dimensional lattice) as in [4]. More precisely, $V(\mathcal{P}) = \mathbb{Z}$ with base vertex 0. The weights are defined as follows:

$$w(i,j) = \begin{cases} \dfrac{1}{2}, & i = 0,\ j = \pm 1, \\ \dfrac{1}{\mu+1}, & i > 0,\ j = i+1, \ \text{ or } i < 0, j = i-1 \\ \dfrac{\mu}{\mu+1}, & i > 0,\ j = i-1, \ \text{ or } i < 0, j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

Let $\mathcal{P}_+$ be the infinite ray with $V(\mathcal{P}_+) = \mathbb{N}$, base vertex 0 and the weights are defined as follows:

$$w(i,j) = \begin{cases} 1, & i = 0,\ j = 1, \\ \dfrac{2}{\mu+1}, & i > 0,\ j = i+1 \\ \dfrac{2\mu}{\mu+1}, & i > 0 \\ 0, & \text{otherwise.} \end{cases}$$

It is a direct calculation to show that the map

$$\pi : V(\mathcal{P}) \to V(\mathcal{P}_+), \quad \pi(i) = |i|$$

is a covering map.

For any $l > 0$, let $\mathcal{P}^{(l)}$ be the subgraph of $\mathcal{P}$ with vertex set $V(\mathcal{P}^{(l)}) = \{0, \pm 1, \ldots, \pm l\}$. The weights are the same as in $\mathcal{P}$ except that

$$w(-l, -l+1) = w(l, l-1) = 1.$$

Also, we define the subgraph $\mathcal{P}_+^{(l)}$ of $\mathcal{P}_+$ with the vertex set $V(\mathcal{P}_+^{(l)}) = \{0, 1, \ldots, l\}$. Again, the weights are the same as in $\mathcal{P}_+$ except that $w(l, l-1) = 2$. Then the map π restricts to a cover from $\mathcal{P}^{(l)}$ to $\mathcal{P}_+^{(l)}$. As in Example 3.1, the Laplacian of $\mathcal{P}^{(l)}$ is given by:

$$\mathcal{L}^{(l)}(i,j) = \begin{cases} 1; & i = j, \\ -\sqrt{\dfrac{1}{k}}; & (i,j) = (0,1) \text{ or } (1,0), \\ -\dfrac{\sqrt{k-1}}{k}; & |i-j|,\ 0 < i, j < l, \\ -\sqrt{\dfrac{k-1}{k}}; & (i,j) = (l-1, l) \text{ or } (l, l-1), \\ 0; & \text{otherwise.} \end{cases}$$

where $k = \frac{\mu+1}{\mu}$. As in Example 3.1, the eigenvalues of $\mathcal{L}^{(l)}$ are 0, 2 and

$$\lambda_n = 1 - \frac{2\sqrt{k-1}}{k} \cos \frac{\pi n}{l} = 1 - \frac{2\sqrt{\mu}}{\mu+1} \cos \frac{\pi n}{l}, \quad \text{for } n = 1, \ldots, l-1.$$

The eigenfunction ϕ_0 associated with the eigenvalue 0 can be written as $f_0/\|f_0\|$ where

$$\begin{aligned} f_0(0) &= 1, \\ f_0(p) &= \sqrt{\frac{\mu+1}{\mu^p}}, \text{ for } 1 \le p \le l-1, \\ f_0(l) &= \frac{1}{\sqrt{\mu^{l-1}}}. \end{aligned}$$

We now consider the eigenfunction ϕ_l which corresponds to the eigenvalue 2. In this case, $\phi_l = f_l/\|f_l\|$, where f_l is defined as follows:

$$\begin{aligned} f_l(0) &= 1, \\ f_l(p) &= (-1)^p\sqrt{\frac{\mu+1}{\mu^p}}, \text{ for } 1 \le p \le l-1, \\ f_l(l) &= (-1)^l\frac{1}{\sqrt{\mu^{l-1}}}. \end{aligned}$$

For each $n \in \{1, 2, \dots, l-1\}$ there is an eigenfunction ϕ_n associated with the eigenvalue μ_n. We can write $\phi_n = f_n/\|f_n\|$ where

$$\begin{aligned} f_n(0) &= \sqrt{\mu+1}\sin\tfrac{\pi n}{l}, \\ f_n(p) &= \sin\tfrac{\pi n(p+1)}{l} - \mu\sin\tfrac{\pi n(p-1)}{l}, \text{ for } 1 \le p \le l-1, \\ f_n(l) &= \sqrt{\mu(\mu+1)}\sin\tfrac{\pi n}{l}. \end{aligned}$$

For $n = 1, \dots, l-1$,

$$\|f_j\|^2 = l(\mu+1)^2\left(1 - \frac{4\mu}{(\mu+1)^2}\cos^2\frac{n\pi}{l}\right).$$

Therefore the heat kernel $h_t^{(l)}$ of $P^{(l)}$ satisfies

$$h_t^{(l)}(0,0) = \sum_{j=1}^{l-1} \frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos\frac{j\pi}{l})}\sin^2\frac{j\pi}{l}}{\frac{l(\mu+1)}{2}(1-\frac{4\mu}{(\mu+1)^2}\cos^2\frac{j\pi}{l})} + \frac{1}{\|f_0\|^2} + \frac{1}{\|f_l\|^2}.$$

Let $H_t^{(l)}$ denote the heat kernel of $\mathcal{P}^{(l)}$. Then, as in Example 4.1, $H_t^{(l)} = H_{1,t}^{(l)} + H_{1,t}^{(l)}$. Since $H_{1,t}^{(l)}(0,a)$ depends only on $|a|$, we have that

Proposition 5.1. *For each $a \in \mathbb{Z}$, the heat kernel H_t of the homesick random walk on the infinite path satisfies*

$$H_{1,t}(0,a) = \frac{\mu\sqrt{2(\mu+1)}}{\pi}\int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos x)}\sin x\left[\frac{1}{\mu}\sin(|a|+1)x - \sin(|a|-1)x\right]}{\mu^2+2\mu+1-4\mu\cos^2 x}dx.$$

Let λ be an eigenvalue of $L^{(l)}$ whose eigenfuctions ϕ vanish at 0. Direct calculations show that:

(1) $\phi(-a) = -\phi(a)$.
(2) λ is an eigenvalue of the matrix

$$\Lambda^{(l)}(i,j) = \begin{cases} 1, & \text{if } i = j, \\ -\dfrac{1}{\sqrt{\mu+1}}, & \text{if } i, j \in \{-l, -(l-1)\}, i \neq j, \\ -\dfrac{\sqrt{\mu}}{\mu+1}, & \text{if } |i-j| = 1, \text{ and not in the last case,} \\ 0, & \text{otherwise} \end{cases}$$

The substitution $r = \mu + 1$, gives the matrix in Section 5 in [3], except the difference at the 2×2-block at the right low corner.

Now the heat kernel of $\mathcal{P}^{(l)}$ has two summands:

$$H_t^{(l)}(a,b) = H_{1,t}^{(l)}(a,b) + H_{2,t}^{(l)}(a,b),$$

where the first summand involves the eigenvalues of the Laplacian with eigenfunction not vanishing at the vertex 0 and the second the eigenvalues with the corresponding eigenfunction being zero at 0. There are l eigenvalues in the second summand, so there are $2l+1$ eigenvalues in the first summand. Since all the eigenvalues in the first summand have multiplicity 1 and their eigenfunctions are their projections to the corresponding eigenfunctions to $\mathcal{P}_+^{(l)}$,

$$H_{1,t}^{(l)}(a,b) = h_t^{(l)}(|a|,|b|).$$

As in Section 5 in [3], taking $l \to \infty$:

$$H_{1,t}^{(l)}(a,b) \quad = h_t^{(l)}(|a|,|b|) =$$

$$\frac{2\mu^2}{\pi} \int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos x)}[\sin((|a|+1)x) - \sin(|a|-1)x][\sin((|b|+1)x) - \sin(|b|-1)x]}{\mu^2 + 2\mu + 1 - 4\mu\cos^2 x} dx.$$

For the calculations of $H_{2,t}(a,b)$, we use the calculations of Section 5 in [3], because as $l \to \infty$, the limit of $\Lambda^{(l)}$ and the Laplacian of the path which is covered by the r-regular tree in [3] are the same. Thus

$$H_{2,t}^{(l)}(a,b) = \frac{2}{\pi} \int_0^\pi \frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos x)}[\sin((|a|+1)x) - \sin(|a|-1)x][\sin((|b|+1)x) - \sin(|b|-1)x]}{\mu^2 + 2\mu + 1 - 4\mu\cos^2 x} dx$$

and the sign is positive if a and b have the same sign and negative otherwise.

Summarizing, we have the following

Theorem 5.2. *For the homesick random walk with parameter μ on the infinite line, the heat kernel $H_t(a,b)$ satisfies:*

(1) *If* $b = 0$, *then* $H_t(a, 0)$ *is*

$$\frac{\mu\sqrt{2(\mu+1)}}{\pi}\int_0^{\pi}\frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos x)}\sin x\left[\frac{1}{\mu}\sin(a+1)x-\sin(a-1)x\right]}{\mu^2+2\mu+1-4\mu\cos^2 x}dx.$$

(2) *If both* a *and* b *are not* 0, *then* $H_t(a, b)$ *is:*

$$\pm\frac{2}{\pi}(\mu^2+\varepsilon)\int_0^{\pi}\frac{e^{-t(1-\frac{2\sqrt{\mu}}{\mu+1}\cos x)}[\sin((|a|+1)x)-\sin(|a|-1)x][\sin((|b|+1)x)-\sin(|b|-1)x]}{\mu^2+2\mu+1-4\mu\text{cos}^2x}dx,$$

where $\varepsilon = sign(a){\cdot}sign(b)$.

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